Vježbe

MA, ETF 19/20

V1

Duborija, Mosurović, Šuković

Linearna algebra i analitička geom. ZZ

Diferencijalni i integralni račun ZZ

Miličić - Uščumlić

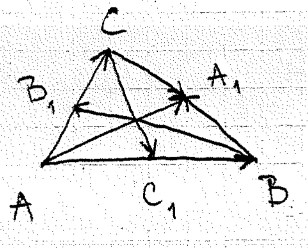
Vektori - orijentisane duži

① Neka su A_1, B_1, C_1 redom sredine stranica BC, CA i AB i neka je $\vec{AB} = \vec{a}$ i $\vec{AC} = \vec{b}$

a) pomoću vektora $\vec{a}$ i $\vec{b}$ izraziti vektore $\vec{AA_1}$, $\vec{BB_1}$ i $\vec{CC_1}$

b) dokazati da je $\vec{AA_1} + \vec{BB_1} + \vec{CC_1} = \vec{0}$

a)



$$\vec{AA_1} = \vec{AB} + \vec{BA_1}$$
$$\vec{AA_1} = \vec{AC} + \vec{CA_1}$$
$$2\vec{AA_1} = \vec{AB} + \vec{AC} + \underbrace{\vec{BA_1} + \vec{CA_1}}_{\vec{0}}$$
$$\vec{AA_1} = \frac{1}{2}(\vec{AB} + \vec{AC}) = \frac{1}{2}(\vec{a} + \vec{b})$$

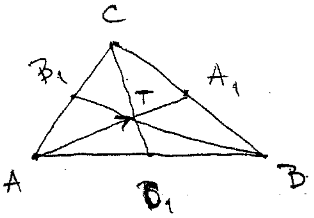
$$\vec{BB_1} = \vec{BA} + \vec{AB_1}$$
$$\vec{BB_1} = -\vec{AB} + \frac{1}{2}\vec{AC} =$$
$$\vec{BB_1} = -\vec{a} + \frac{1}{2}\vec{b}$$

$$\vec{CC_1} = \frac{1}{2}(\vec{CA} + \vec{CB})$$
$$\vec{CC_1} = \frac{1}{2}\vec{CA} + \frac{1}{2}\vec{CB}$$
$$= -\frac{1}{2}\vec{AC} + \frac{1}{2}(\vec{CA} + \vec{AB})$$
$$= -\frac{1}{2}\vec{AC} + \frac{1}{2}(-\vec{AC} + \vec{AB})$$
$$= -\frac{1}{2}\vec{b} - \frac{1}{2}\vec{b} + \frac{1}{2}\vec{a}$$
$$= \frac{1}{2}\vec{a} - \vec{b}$$

b) $\vec{AA_1} + \vec{BB_1} + \vec{CC_1} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b} - \vec{a} + \frac{1}{2}\vec{b} + \frac{1}{2}\vec{a} - \vec{b} =$

$= \vec{0} + \vec{0} = \vec{0}$

② Neka su A, B i C tjemena, a T težište $\triangle ABC$. Dokazati da je $\vec{AT} + \vec{BT} + \vec{CT} = \vec{0}$



$$AT : TA_1 = 2 : 1$$

$$\vec{AT} = \frac{2}{3}\vec{AA_1}$$

$$\vec{AT} = \frac{2}{3}\cdot\frac{1}{2}(\vec{AB}+\vec{AC}) = \frac{1}{3}(\vec{AB}+\vec{AC}) = \frac{1}{3}\vec{AB}+\frac{1}{3}\vec{AC}$$

$$\vec{BT} = \frac{2}{3}\vec{BB_1} = \frac{2}{3}\cdot\frac{1}{2}(\vec{BA}+\vec{BC}) = \frac{1}{3}(-\vec{AB}+\vec{BA}+\vec{AC}) =$$

$$= \frac{1}{3}(-2\vec{AB}+\vec{AC}) = -\frac{2}{3}\vec{AB}+\frac{1}{3}\vec{AC}$$

$$\vec{CT} = \frac{2}{3}\vec{CC_1} = \frac{2}{3}\cdot\frac{1}{2}(\vec{CA}+\vec{CB}) = \frac{1}{3}(-\vec{AC}+\vec{CA}+\vec{AB}) =$$

$$= -\frac{2}{3}\vec{AC}+\frac{1}{3}\vec{AB}$$

$$\vec{AT}+\vec{BT}+\vec{CT} = \underline{\frac{1}{3}\vec{AB}} + \underline{\underline{\frac{1}{3}\vec{AC}}} \underline{-\frac{2}{3}\vec{AB}} + \underline{\underline{\frac{1}{3}\vec{AC}}} \underline{\underline{-\frac{2}{3}\vec{AC}}} + \underline{\frac{1}{3}\vec{AB}} = \vec{0}$$

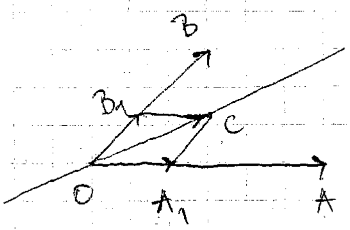
Def. Vektori $\vec{a}$ i $\vec{b}$ su kolinearni ako leže na istoj ili paralelnim pravama.

T: Vektori $\vec{a}$ i $\vec{b}$ su kolinearni akko $\vec{a} = k\cdot\vec{b}$ (za $k \neq 0$)

Def. Vektori $\vec{a}, \vec{b}, \vec{c}$ su komplanarni ako leže u istoj ili paralelnim ravnima

T: Vektori $\vec{a}, \vec{b}, \vec{c}$ su komplanarni akko je $\vec{c} = k\vec{a} + m\vec{b}$

(3) Odrediti vektor pravca simetrale ugla $\measuredangle AOB$, gdje je $\vec{OA} = \vec{a}$ i $\vec{OB} = \vec{b}$



$\vec{OA_1}$ - jedinični vektor vektora $\vec{OA}$

$$\vec{OA_1} = \frac{1}{|\vec{a}|}\cdot\vec{a}$$

$\vec{OB_1}$ - jedinični vektor vektora $\vec{OB}$

$$\vec{OB_1} = \frac{1}{|\vec{b}|}\cdot\vec{b}$$

$\vec{x}$

$\vec{x_0}$ - jed. vektor vektora $\vec{x}$

$$\vec{x_0} = \frac{1}{|\vec{x}|}\cdot\vec{x}$$

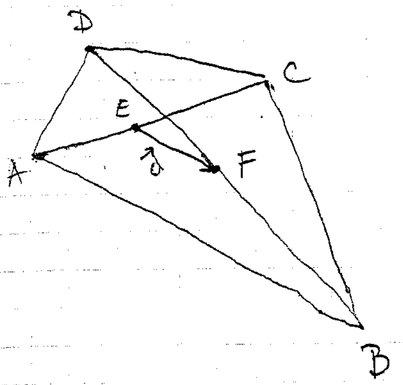
$$\vec{x} = |\vec{x}|\cdot\vec{x_0}$$

Nad vektorima $\vec{OA_1}$ i $\vec{OB_1}$ konstruišemo romb OA_1CB_1

Vektor dijagonale $\vec{OC}$ polovi unutrašnji ugao romba, tj. $\measuredangle A_1OB_1$, a samim tim i $\measuredangle AOB$. Vektor $\vec{OC}$ je vektor pravca simetrale $\measuredangle AOB$.

$$\vec{OC} = \vec{OA_1} + \vec{A_1C} = \vec{OA_1} + \vec{OB_1} = \frac{1}{|\vec{a}|}\cdot\vec{a} + \frac{1}{|\vec{b}|}\cdot\vec{b}$$

④ Dokazati da je u četvorouglu ABCD vektor $\vec{d}$ koji spaja sredine dijagonala AC i BD dat sa $\vec{d} = \frac{1}{2}(\vec{AD} - \vec{BC})$



E – središte dijagonale AC

F – —— || —— BD

$$\vec{d} = \vec{EF}$$

$$\vec{EF} = \vec{EA} + \vec{AD} + \vec{DF}$$

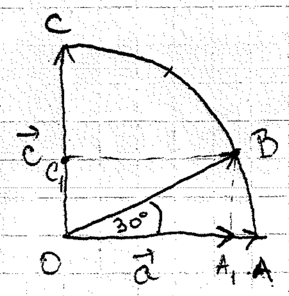
$$\vec{EF} = \vec{EC} + \vec{CB} + \vec{BF}$$

$$2\vec{EF} = \underbrace{\vec{EA} + \vec{EC}}_{\vec{0}} + \vec{AD} + \vec{CB} + \underbrace{\vec{DF} + \vec{BF}}_{\vec{0}}$$

$$\vec{EF} = \frac{1}{2}(\vec{AD} + \vec{CB})$$

$$\vec{EF} = \frac{1}{2}(\vec{AD} - \vec{BC})$$

⑤ Tačka B dijeli luk $\overset{\frown}{AC}$ nad uglom od $90°$ u odnosu $1:2$. Razložiti vektor $\vec{OC} = \vec{c}$ duž vektora $\vec{OA} = \vec{a}$ i $\vec{OB} = \vec{b}$, gdje je O centar kruga.



A_1 – ortogonalna projekcija tačke B na $p(O, A)$

C_1 – —— || —— na $q(OC)$

$$\measuredangle AOB = \measuredangle A_1OB = 30°$$

$\vec{OA_1}$ – vektorska projekcija vektora $\vec{OB}$ na p

$|\vec{OA_1}|$ – skalarna —— || ——

$|\vec{OA_1}| = \cos 30° \cdot |\vec{OB}|$ Uočimo $|\vec{OA}| = |\vec{OB}| = |\vec{OC}| = r$

$|\vec{OA_1}| = \frac{\sqrt{3}}{2} |\vec{OA}|$ (*)

Vektori $\vec{OA_1}$ i $\vec{OA}$ su istog pravca i smjera pa iz (*) slijedi da je $\vec{OA_1} = \frac{\sqrt{3}}{2}\vec{OA} = \frac{\sqrt{3}}{2}\vec{a}$

$\triangle OA_1B:\quad \sin 30^\circ = \dfrac{|\vec{A_1B}|}{|\vec{OB}|} = \dfrac{|\vec{OC_1}|}{|\vec{OB}|}$

$|\vec{OC_1}| = \frac{1}{2}|\vec{OB}|$

$|\vec{OC_1}| = \frac{1}{2}|\vec{OC}|$

$\vec{OC_1} = \frac{1}{2}\vec{OC}$

$\vec{OC_1} = \frac{1}{2}\vec{c}$

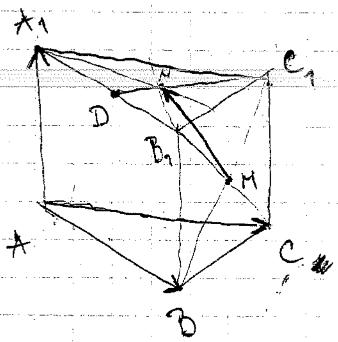
$\vec{OB} = \vec{OA_1} + \vec{A_1B}$

$\vec{OB} = \vec{OA_1} + \vec{OC_1}$

$\vec{b} = \frac{\sqrt{3}}{2}\vec{a} + \frac{1}{2}\vec{c}$

$\vec{c} = -\sqrt{3}\,\vec{a} + 2\vec{b}$

(6) Data je prostorna prizma $ABCA_1B_1C_1$. Tačka M je centar paralelograma BCC_1B_1, a tačka N težište $\triangle A_1B_1C_1$. Izraziti vektor $\vec{MN}$ preko vektora $\vec{AB}$, $\vec{AC}$ i $\vec{AA_1}$.



D - središte duži A_1B_1

$\vec{MN} = \vec{MC_1} + \vec{C_1N}$

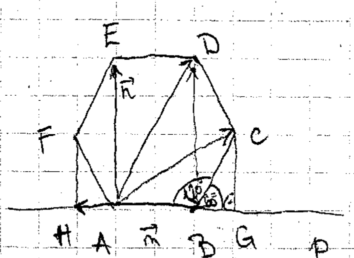
$\vec{MC_1} = \frac{1}{2}\vec{BC_1} = \frac{1}{2}(\vec{BC} + \vec{CC_1}) = \frac{1}{2}(\vec{AC} - \vec{AB} + \vec{AA_1}) =$

$= -\frac{1}{2}\vec{AB} + \frac{1}{2}\vec{AC} + \frac{1}{2}\vec{AA_1}$

$\vec{C_1N} = \frac{2}{3}\vec{C_1D} = \frac{2}{3}\cdot\frac{1}{2}(\vec{C_1A_1} + \vec{C_1B_1}) = \frac{1}{3}(-\vec{AC} + \vec{AB} - \vec{AC}) = \frac{1}{3}\vec{AB} - \frac{2}{3}\vec{AC}$

$\vec{MN} = \underline{-\frac{1}{2}\vec{AB}} + \underline{\underline{\frac{1}{2}\vec{AC}}} + \frac{1}{2}\vec{AA_1} + \underline{\frac{1}{3}\vec{AB}} \underline{\underline{-\frac{2}{3}\vec{AC}}} = -\frac{1}{6}\vec{AB} - \frac{1}{6}\vec{AC} + \frac{1}{2}\vec{AA_1}$

(7) U pravilnom šestouglu $ABCDEF$ dati su vektori $\vec{AB} = \vec{m}$ i $\vec{AE} = \vec{n}$. Izraziti vektore $\vec{AC}$, $\vec{AD}$, $\vec{AF}$ i $\vec{EF}$ preko vektora $\vec{m}$ i $\vec{n}$.



G - projekcija tačke C na pravu $p(AB)$

$\sphericalangle ABC = 120^\circ \Rightarrow \sphericalangle GBC = 60^\circ$

$\triangle BGC:\quad \cos 60^\circ = \dfrac{|\vec{BG}|}{|\vec{BC}|}$

$|\vec{BG}| = \frac{1}{2}|\vec{BC}| = \frac{1}{2}|\vec{AB}| \Rightarrow \vec{BG} = \frac{1}{2}\vec{AB} = \frac{1}{2}\vec{m}$

$\vec{GC} = \frac{1}{2}|\vec{AE}| = \frac{1}{2}\vec{n}$

$\vec{AD} = \vec{AB} + \vec{BD}$

$\boxed{\vec{AD} = \vec{m} + \vec{n}}$

$\vec{AC} = \vec{AB} + \vec{BG} + \vec{GC}$

$\vec{AC} = \vec{m} + \frac{1}{2}\vec{m} + \frac{1}{2}\vec{n}$

$\boxed{\vec{AC} = \frac{3}{2}\vec{m} + \frac{1}{2}\vec{n}}$

H-projekcija F na p

$\vec{AH} = -\vec{BG} = -\frac{1}{2}\vec{m}$

$\vec{AF} = \vec{AH} + \vec{HF}$

$\vec{AF} = -\frac{1}{2}\vec{m} + \frac{1}{2}\vec{n}$

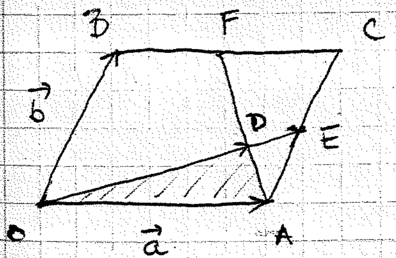
$\vec{EF} = \vec{AF} - \vec{AE}$

$\vec{EF} = -\frac{1}{2}\vec{m} + \frac{1}{2}\vec{n} - \vec{n}$

$\boxed{\vec{EF} = -\frac{1}{2}\vec{m} - \frac{1}{2}\vec{n}}$

(8) Dat je paralelogram OACB. Neka je E središte stranice AC, a F središte stranice CB. Neka je D presjek duži OE i A

Odrediti $|\vec{AD}| : |\vec{AF}|$ i $\frac{|\vec{OD}|}{|\vec{OE}|}$



$\vec{OD} = k \cdot \vec{OE} = k(\vec{OA} + \vec{AE}) = k(\vec{a} + \frac{1}{2}\vec{b})$

$\vec{AD} = m\,\vec{AF} = m(\vec{AC} + \vec{CF}) = m(\vec{b} + \frac{1}{2}(-\vec{a})) = m(\vec{b} - \frac{1}{2}$

$\vec{OA} + \vec{AD} = \vec{OD}$

$\vec{a} + m(\vec{b} - \frac{1}{2}\vec{a}) = k(\vec{a} + \frac{1}{2}\vec{b})$

$(1 - \frac{m}{2} - k)\vec{a} + (m - \frac{k}{2})\vec{b} = \vec{0} \quad (*)$

Kako su vektori $\vec{a}$ i $\vec{b}$ nekolinear to iz $(*)$ slijedi da je

$$\begin{cases} 1 - \frac{m}{2} - k = 0 \\ m - \frac{k}{2} = 0 \end{cases} \Big) \cdot 2$$

$1 - \frac{m}{2} - k = 0$

$2 - \frac{5}{2}k = 0$

$\boxed{k = \frac{4}{5}}$

$\Downarrow$

$\boxed{m = \frac{2}{5}}$

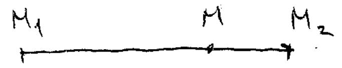
$\vec{OD} = \frac{4}{5}\vec{OE}$

$\vec{AD} = \frac{2}{5}\vec{AF}$

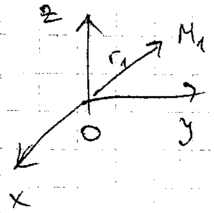
$\frac{|\vec{OD}|}{|\vec{OE}|} = \frac{4}{5}$

$\frac{|\vec{AD}|}{|\vec{AF}|} = \frac{2}{5}$

(9)



$M_1(\vec{r_1}),\ M_2(\vec{r_2}),\ M(\vec{r})$



Tačka M dijeli duž M_1M_2 u omjeru $|\overrightarrow{M_1M}| : |\overrightarrow{MM_2}| = |\lambda|$, $\lambda \neq -1$

akko $\vec{r} = \frac{1}{1+\lambda}(\vec{r_1} + \lambda \vec{r_2})$

$M_1(x_1, y_1, z_1),\ M_2(x_2, y_2, z_2),\ M_3(x_3, y_3, z_3)$

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}$$

$$y = \frac{y_1 + \lambda y_2}{1+\lambda}$$

$$z = \frac{z_1 + \lambda z_2}{1+\lambda}$$

Ako je $\lambda = 1$, tačka M je središte duži

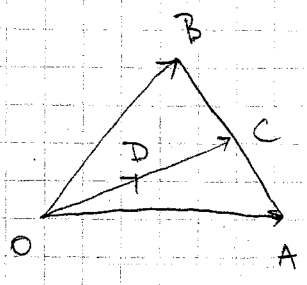
$$x = \frac{x_1 + x_2}{2},\quad y = \frac{y_1 + y_2}{2},\quad z = \frac{z_1 + z_2}{2}$$

3. oktobar

1. Neka su $A(-1,-2,-4)$, $B(-4,-2,0)$, $O(3,-2,1)$ tjemena trougla. Tačka C dijeli duž AB u omjeru $3:4$, a tačka D dijeli duž OC u omjeru ~~...~~ $2:5$.

a) Izračunati vektor $\overrightarrow{OD}$ preko vektora $\overrightarrow{OA}$ i $\overrightarrow{OB}$

b) Naći koordinate tačke C



$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OA} + \frac{3}{7}\cdot\overrightarrow{AB} = \overrightarrow{OA} + \frac{3}{7}(\overrightarrow{OB} - \overrightarrow{OA}) =$$

$$= \frac{4}{7}\overrightarrow{OA} + \frac{3}{4}\overrightarrow{OB}$$

$$\overrightarrow{OD} = \frac{2}{7}\cdot\overrightarrow{OC} = \frac{2}{7}\left(\frac{4}{7}\overrightarrow{OA} + \frac{3}{7}\overrightarrow{OB}\right) = \frac{8}{49}\overrightarrow{OA} + \frac{6}{49}\overrightarrow{OB}$$

$A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$

$M(x, y, z)$ = središte duži AB

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2}$$

$$|AC| = |CB| = \frac{3}{4}$$

$C(x, y, z)$ $\quad x = -\frac{16}{7}, \; y = -2, \; z = -\frac{16}{7}$ $\quad C\left(-\frac{16}{7}, -2, -\frac{16}{7}\right)$

Skalarni proizvod vektora

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cdot \cos \measuredangle(\vec{a}, \vec{b})$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| \, pr_{\vec{a}} \vec{b}$$

$$pr_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \qquad pr_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

$\vec{a} = (x_1, y_1, z_1)$

$\vec{b} = (x_2, y_2, z_2)$

$$\vec{a} \cdot \vec{b} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

<u>Pr</u> Ako je $|\vec{a}| = 3$, $|\vec{b}| = 4$, $\measuredangle(\vec{a}, \vec{b}) = \frac{2\pi}{3}$, naći

a) $\vec{a} \cdot \vec{b}$

b) $(3\vec{a} - 2\vec{b})(\vec{a} + 2\vec{b})$

c) $|\vec{a} + \vec{b}|^2$

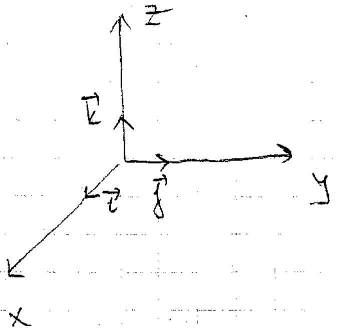
a) $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \frac{2\pi}{3} = 3 \cdot 4 \cdot \left(-\frac{1}{2}\right) = -6$

b) $(3\vec{a} - 2\vec{b})(\vec{a} + 2\vec{b}) = 3\vec{a} \cdot \vec{a} + 6\vec{a} \cdot \vec{b} - 2\vec{b} \cdot \vec{a} - 4\vec{b} \cdot \vec{b} =$

$= 3|\vec{a}|^2 + 4\vec{a} \cdot \vec{b} - 4|\vec{b}|^2 = 3 \cdot 9 + 4 \cdot (-6) - 4 \cdot 16 = 27 - 24 - 64 = -61$

c) $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})(\vec{a} + \vec{b}) = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = 9 + 2 \cdot (-6) + 16 = 13$

Pr Naći skalarni proizvod vektora $\vec{a} = (3,4,7)$, $\vec{b} = (2,-5,2)$



$$\vec{a} \cdot \vec{b} = 3 \cdot 2 + 4 \cdot (-5) + 7 \cdot 2 = 0$$

Primijetimo da je $\vec{a} \perp \vec{b}$

1. Naći vektor $\vec{c}$ koji je kolinearan sa vektorom $\vec{a} + \vec{b}$ ako su ispunjeni uslovi $\vec{a} \cdot \vec{b} = 5$, $\vec{c} \cdot \vec{b} = 18$, $|\vec{b}| = 2$

$\vec{c} = k(\vec{a} + \vec{b})$

I $\quad k(\vec{a} + \vec{b}) \cdot \vec{b} = 18$

II $\quad \vec{c} = k(\vec{a} + \vec{b}) \;/\cdot \vec{b}$

$$\vec{c} \cdot \vec{b} = k\vec{a} \cdot \vec{b} + k \cdot \vec{b} \cdot \vec{b}$$

I $\quad k\vec{a} \cdot \vec{b} + k|\vec{b}|^2 = 18$

$$5k + 4k = 18$$

$$9k = 18$$

$$k = 2$$

$$\vec{c} = 2(\vec{a} + \vec{b})$$

2. Naći ugao između vektora $\vec{a}$ i $\vec{b}$ ako je $|\vec{a}| = 2|\vec{b}|$ i vektor $2\vec{a} + \vec{b}$ je normalan na vektor $\vec{a} - 3\vec{b}$

$$(2\vec{a} + \vec{b}) \perp (\vec{a} - 3\vec{b})$$

$$(2\vec{a} + \vec{b})(\vec{a} - 3\vec{b}) = 0$$

$$2|\vec{a}|^2 - 5\vec{a} \cdot \vec{b} - 3|\vec{b}|^2 = 0$$

$$8|\vec{b}|^2 - 5\vec{a} \cdot \vec{b} - 3|\vec{b}|^2 = 0$$

$$|\vec{b}|^2 - \vec{a} \cdot \vec{b} = 0$$

$$\vec{a} \cdot \vec{b} = |\vec{b}|^2$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\varphi$$

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

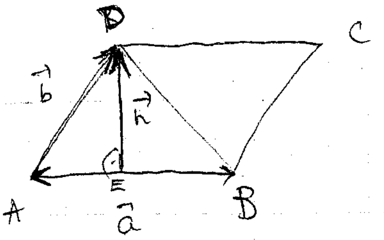
$$\cos\varphi = \frac{|\vec{b}|^{\cancel{2}}}{2\cancel{|\vec{b}|}\,\cancel{|\vec{b}|}}$$

$$\cos\varphi = \frac{1}{2}$$

$$\boxed{\varphi = \frac{\pi}{3}}$$

$\vec{a} : \vec{b}$

3. Dokazati da je vektor $\vec{h}$ visine paralelograma nad vektorima $\vec{a}$ i $\vec{b}$ dat sa $\vec{h} = \vec{b} - \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\vec{a}_0$ gdje je $\vec{a}_0$ jedinični vektor vektora $\vec{a}$



$\vec{AB} = \vec{a}$, $\vec{AD} = \vec{b}$

E – podnožje visine iz tjemena B na osnovicu AB

$\vec{h} = \vec{ED} = \vec{EA} + \vec{AD}$

$\vec{EA} = |\vec{EA}|\cdot\vec{e}_0$, jedinični vektor vektora $\vec{EA}$

$\vec{h} = -\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\vec{a}_0 + \vec{b}$

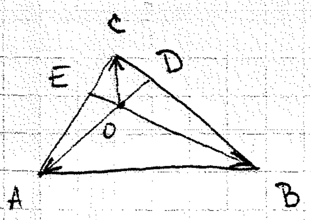
Vektori $\vec{EA}$ i $\vec{AB}$ su kolinearni, ali suprotnog smjera pa su njihovi jedinični vektori suprotni $\vec{e}_0 = -\vec{a}_0$

$$\boxed{\vec{h} = \vec{b} - \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\vec{a}_0}$$

Uočimo da je $|\vec{EA}| = |\vec{AE}| = pr_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

$\vec{EA} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot(-\vec{a}_0)$

4. Dokazati da se u svakom trouglu visine sijeku u jednoj tački.



$\vec{AD}$ – visina iz A

BE – visina iz B

O – presjek visina AD i BE

Dokažimo da je $CO \perp AB$ i time ćemo pokazati da visina iz tjemena C prolazi kroz tačku O.

$\vec{OA} \perp \vec{BC}$

$\vec{OB} \perp \vec{CA}$

$\vec{OA}\cdot(\vec{OC} - \vec{OB}) = 0$

$\vec{OB}\cdot(\vec{OA} - \vec{OC}) = 0$

$\vec{OA}\cdot\vec{OC} - \vec{OA}\cdot\vec{OB} = 0 \quad (1)$

$\vec{OB}\cdot\vec{OA} - \vec{OB}\cdot\vec{OC} = 0 \quad (2)$

$(1)+(2): \quad \vec{OA}\cdot\vec{OC} - \vec{OB}\cdot\vec{OC} = 0$

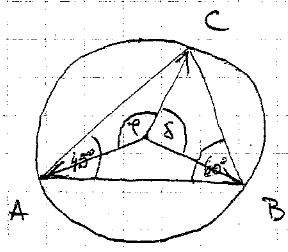
$\vec{BA}\cdot\vec{OC} = 0 \Rightarrow \vec{BA} \perp \vec{OC}$

$(\vec{OA} - \vec{OB})\cdot\vec{OC} = 0$

(*) Naći ugao koji obrazuju vektori $\vec{a}$ i $\vec{b}$ ako je $(\vec{a}-2\vec{b}) \perp (\vec{a}+3\vec{b})$ i $(\vec{a}+\vec{b}) \perp (2\vec{a}+\vec{b})$

5. U $\triangle ABC$ dati su uglovi $\alpha = 45^\circ$ i $\beta = 60^\circ$. Oko trougla opisan je krug poluprečnika $r = 1$ sa centrom u tački O. Odrediti:

a) $\vec{OB} \cdot \vec{OC}$, b) $\vec{OC} \cdot \vec{OA}$



$|\vec{OA}| = |\vec{OB}| = |\vec{OC}| = r = 1$

$\sphericalangle(\vec{OB}, \vec{OC}) = 2 \cdot 45^\circ = 90^\circ$

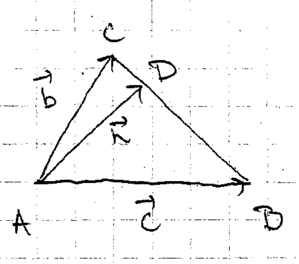
Centralni ugao dvostruko veći od periferijskog nad istim lukom

$\vec{OB} \cdot \vec{OC} = |\vec{OB}||\vec{OC}| \cos 90^\circ = 0$

b) $\sphericalangle(\vec{OA}, \vec{OC}) = 2 \cdot 60^\circ = 120^\circ$

$\vec{OA} \cdot \vec{OC} = |\vec{OA}||\vec{OC}| \cos 120^\circ = 1 \cdot 1 \cdot (-\frac{1}{2}) = -\frac{1}{2}$

6. $\triangle ABC$ zadat je vektorima $\vec{AB} = \vec{c}$ i $\vec{AC} = \vec{b}$. Izraziti pomoću vektora $\vec{c}$ i $\vec{b}$ vektor visine iz tjemena A.



$\vec{h} = \vec{AD}$

$\vec{h} = \vec{AB} + \vec{BD}$

$\vec{h} = \vec{c} + k \cdot \vec{BC}$

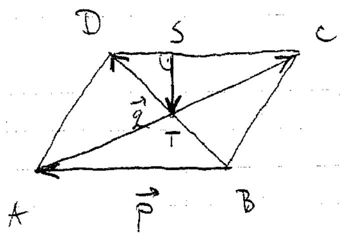
$\vec{h} = \vec{c} + k(\vec{b} - \vec{c}) \quad / \cdot \vec{BC} = \vec{b} - \vec{c}$

$0 = \vec{c}(\vec{b} - \vec{c}) + k|\vec{b} - \vec{c}|^2$

$k = -\dfrac{\vec{c}(\vec{b} - \vec{c})}{|\vec{b} - \vec{c}|^2}$

$\vec{h} = \vec{c} - \dfrac{\vec{c}(\vec{b} - \vec{c})}{|\vec{b} - \vec{c}|^2} \cdot (\vec{b} - \vec{c})$

7. Neka je T presječna tačka dijagonala paralelograma $ABCD$ i neka je S podnožje visine $\triangle TCD$ iz tjemena T. Izraziti vektor $\vec{ST}$ u bazi $\vec{BA} = \vec{p}$ i $\vec{BD} = \vec{q}$



$\vec{ST} = -\vec{TS}$

$$\vec{ST} = \vec{SD} + \vec{DT}$$

$$\vec{ST} = k \cdot \vec{CD} + \frac{1}{2}\vec{DB}$$

$$\vec{ST} = k \cdot (+\vec{p}) + \frac{1}{2} \cdot (-\vec{q}) \quad / \cdot \vec{p}$$

$$0 = k \cdot |\vec{p}|^2 - \frac{1}{2}\vec{p} \cdot \vec{q}$$

$$k = \frac{1}{2} \cdot \frac{\vec{p} \cdot \vec{q}}{|\vec{p}|^2}$$

$$\vec{ST} = \frac{1}{2} \cdot \frac{\vec{p} \cdot \vec{q}}{|\vec{p}|^2} \cdot \vec{p} - \frac{1}{2}\vec{q}$$

Vektorski proizvod

Pr Ako je $\vec{a} = (3, -1, 2)$ i $\vec{b} = (1, 2, -1)$ naći : a) $\vec{a} \times \vec{b}$, b) $(2\vec{a} + \vec{b}) \times \vec{b}$

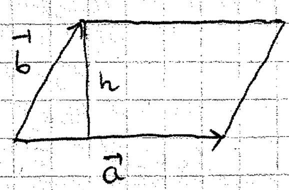
c) $(2\vec{a} - \vec{b}) \times (2\vec{a} + \vec{b})$

a) $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} \cdot \vec{j} + \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} \vec{k} =$$

$$= (1-4)\vec{i} - (-3-2) \cdot \vec{j} + (6-(-1))\vec{k} = -3\vec{i} + 5\vec{j} + 7\vec{k}$$

$$\vec{a} \times \vec{b} = (-3, 5, 7)$$

P - površina paralelograma nad $\vec{a}$ i $\vec{b}$



$$P = |\vec{a} \times \vec{b}|$$

$$P = \sqrt{9+25+49} = \sqrt{83}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \measuredangle(\vec{a}, \vec{b})$$

$$\left.\begin{matrix} P = |\vec{a} \times \vec{b}| \\ P = |\vec{a}| h \end{matrix}\right\} \quad h = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$$

$$h = \frac{\sqrt{83}}{\sqrt{9+1+4}} = \frac{\sqrt{83}}{\sqrt{14}}$$

b) $(2\vec{a}+\vec{b})\times\vec{b} = 2\vec{a}\times\vec{b} + \underbrace{\vec{b}\times\vec{b}}_{\vec{0}} = 2\vec{a}\times\vec{b} = 2(-3,5,7) = (-6,10,14)$

c) $(2\vec{a}-\vec{b})\times(2\vec{a}+\vec{b}) = 4\underbrace{\vec{a}\times\vec{a}}_{\vec{0}} + 2\vec{a}\times\vec{b} - \vec{b}\times(2\vec{a}) - \underbrace{\vec{b}\times\vec{b}}_{\vec{0}} = 2\vec{a}\times\vec{b} + 2\vec{a}\times\vec{b} =$

$= 4\,\vec{a}\times\vec{b} = 4(-3,5,7) = (-12,20,28)$

2. Za koju vrijednost parametra k su vektori $\vec{p}=k\vec{a}-5\vec{b}$ i $\vec{q}=3\vec{a}-\vec{b}$ kolinearni ako vektori $\vec{a}$ i $\vec{b}$ nisu kolinearni.

$\vec{p}$ i $\vec{q}$ kolinearni

$\vec{p}\times\vec{q}=\vec{0}$

$(k\vec{a}-5\vec{b})\times(3\vec{a}-\vec{b})=\vec{0}$

$3k\underbrace{\vec{a}\times\vec{a}}_{\vec{0}} - k\vec{a}\times\vec{b} - 15\vec{b}\times\vec{a} + 5\underbrace{\vec{b}\times\vec{b}}_{\vec{0}} = \vec{0}$

$-k\vec{a}\times\vec{b} + 15\vec{a}\times\vec{b} = \vec{0}$

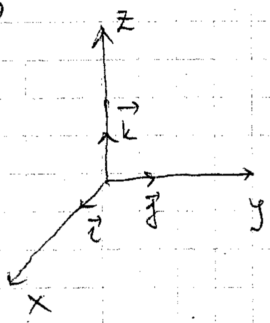
$(15-k)\underbrace{\vec{a}\times\vec{b}}_{\neq\vec{0}} = \vec{0} \quad \Rightarrow \quad \boxed{k=15}$

$\vec{i} \quad \vec{j} \quad \vec{k} \quad \vec{i} \quad \vec{j}$

$\vec{i}\times\vec{i}=\vec{0}$

$\vec{j}\times\vec{j}=\vec{0}$

$\vec{k}\times\vec{k}=\vec{0}$



$\vec{i}\times\vec{j}=\vec{k}$

$\vec{j}\times\vec{k}=\vec{i}$

$\vec{k}\times\vec{i}=\vec{j}$

3. Razložiti vektor $\vec{p}=(3\vec{a}+\vec{b}-2\vec{c})\times(\vec{a}-\vec{b}+5\vec{c})$ u ortonormiranoj bazi $\vec{a},\vec{b},\vec{c}$

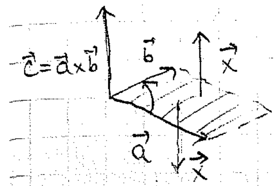
$\vec{a} \quad \vec{b} \quad \vec{c} \quad \vec{a} \quad \vec{b}$

$\vec{p} = -3\vec{c} - 15\vec{b} - \vec{c} + 5\vec{a} - 2\vec{b} - 2\vec{a} = 3\vec{a} - 17\vec{b} - 4\vec{c}$

$\vec{p}=(3,-17,-4)$ u bazi $\{\vec{a},\vec{b},\vec{c}\}$

4. Vektor $\vec{x}$ je ortogonalan na vektore $\vec{a}=(4,-2,-3)$ i $\vec{b}=(0,1,3)$, a sa Oy osom obrazuje tupi ugao. Naći njegove koordinate ako je $|\vec{x}|=26$

$$\left.\begin{array}{l}\vec{x}\perp\vec{a}\\ \vec{x}\perp\vec{b}\end{array}\right\} \Rightarrow \vec{x}=\lambda(\vec{a}\times\vec{b})$$



$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -2 & -3\\ 0 & 1 & 3\end{vmatrix}=\begin{vmatrix}-2 & -3\\ 1 & 3\end{vmatrix}\vec{i}-\begin{vmatrix}4 & -3\\ 0 & 3\end{vmatrix}\vec{j}+\begin{vmatrix}4 & -2\\ 0 & 1\end{vmatrix}\vec{k}=$$

$$=-3\vec{i}-12\vec{j}+4\vec{k}$$

$$\vec{a}\times\vec{b}=(-3,-12,4)$$

$$\vec{x}=(-3\lambda,-12\lambda,4\lambda)$$

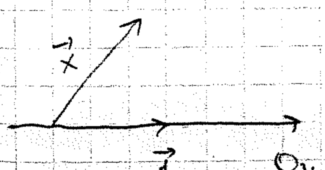
$$|\vec{x}|=26 \Rightarrow \sqrt{9\lambda^2+144\lambda^2+16\lambda^2}=26$$

$$\sqrt{169\lambda^2}=26$$

$$|13\lambda|=26$$

$$13|\lambda|=26$$

$$|\lambda|=2$$



$$\cos\alpha=\frac{\vec{x}\cdot\vec{j}}{|\vec{x}||\vec{j}|}$$

$$\alpha=\sphericalangle(\vec{x},Oy)$$

$$\alpha=\sphericalangle(\vec{x},\vec{j})$$

$$\left.\begin{array}{c}\cos\alpha=\dfrac{-12\lambda}{26\cdot 1}\\ \alpha\text{ - tup ugao}\\ \cos\alpha<0\end{array}\right\}\Rightarrow \underbrace{\lambda>0 \;\Rightarrow\; |\lambda|=2}$$

$$\Downarrow$$

$$\lambda=2$$

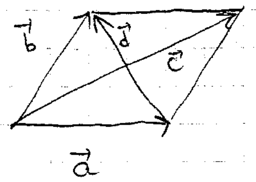
$\vec{j}=(0,1,0)$

$\vec{i}=(1,0,0)$

$\vec{k}=(0,0,1)$

$$\boxed{\vec{x}=(-6,-24,8)}$$

5. Odrediti površinu paralelograma ako su vektori dijagonala $\vec{c} = 3\vec{m} + 3\vec{n}$ i $\vec{d} = \vec{m} - \vec{n}$, $\sphericalangle(\vec{m}, \vec{n}) = \frac{\pi}{6}$, a $\vec{m}$ i $\vec{n}$ su jedinični vektori



$$\vec{a} + \vec{b} = \vec{c}$$
$$\vec{b} - \vec{a} = \vec{d}$$
$$2\vec{b} = \vec{c} + \vec{d}$$
$$\vec{b} = \frac{1}{2}(\vec{c} + \vec{d})$$
$$\vec{b} = \frac{1}{2}(3\vec{m} + 3\vec{n} + \vec{m} - \vec{n}) = \underline{2\vec{m} + \vec{n}}$$

$$\vec{a} = \vec{c} - \vec{b}$$
$$\vec{a} = \vec{c} - \frac{1}{2}\vec{c} - \frac{1}{2}\vec{d} = \frac{1}{2}(\vec{c} - \vec{d})$$
$$= \frac{1}{2}(3\vec{m} + 3\vec{n} - \vec{m} + \vec{n}) =$$
$$= \underline{\vec{m} + 2\vec{n}}$$

$$P = |\vec{a} \times \vec{b}| = |(\vec{m} + 2\vec{n}) \times (2\vec{m} + \vec{n})| = |\vec{m} \times \vec{n} + 4\vec{n} \times \vec{m}| = |\vec{m} \times \vec{n} - 4\vec{m} \times \vec{n}| =$$

$$= |-3\vec{m} \times \vec{n}| = |-3| \cdot |\vec{m} \times \vec{n}| = 3|\vec{m}||\vec{n}| \sin\frac{\pi}{6} = 3 \cdot 1 \cdot 1 \cdot \frac{1}{2} = \frac{3}{2}$$

(*) Odrediti sinus ugla između dijagonala paralelograma konstruisanog nad vektorima $\vec{a} = 2\vec{m} + \vec{n} - \vec{p}$, $\vec{b} = \vec{m} - 3\vec{n} + \vec{p}$ gdje je $\{\vec{m}, \vec{n}, \vec{p}\}$ ortonomirana baza

Miješoviti proizvod

$$(\vec{a} \times \vec{b}) \cdot \vec{c}$$

Odrediti parametar t iz uslova da su vektori $\vec{a}, \vec{b}$ i $\vec{c}$ komplanarni ako je $\vec{a} = (\ln(t-2), -2, 6)$, $\vec{b} = (t, -2, 5)$ i $\vec{c} = (0, -1, 3)$

$$\begin{vmatrix} \ln(t-2) & -2 & 6 \\ t & -2 & 5 \\ 0 & -1 & 3 \end{vmatrix} = 0$$

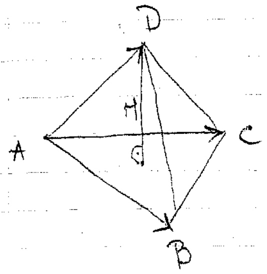
$$\begin{vmatrix} -2 & 5 \\ -1 & 3 \end{vmatrix} \ln(t-2) - \begin{vmatrix} t & 5 \\ 0 & 3 \end{vmatrix} (-2) + \begin{vmatrix} t & -2 \\ 0 & -1 \end{vmatrix} \cdot 6 = 0$$

$$-\ln(t-2) + \cancel{6t} - \cancel{6t} = 0$$

$$-\ln(t-2) = 0$$
$$\ln(t-2) = 0$$
$$t - 2 = 1$$
$$t = 3$$

10. oktobar

1. Tjemena tetraedra su $A(1,1,1)$, $B(0,2,1)$, $C(-1,-2,3)$, $D(3,4,-3)$. Odrediti dužinu visine iz tjemena D.



$$V_t = \frac{1}{6} \left| (\vec{AB} \times \vec{AC}) \cdot \vec{AD} \right| \quad (1)$$

$$V_t = \frac{1}{3} BH, \quad B - \text{površina } \triangle ABC$$

$$V_t = \frac{1}{3} \cdot \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right| \cdot H \quad (2)$$

Iz (1) i (2) slijedi: $H = \dfrac{\left| (\vec{AB} \times \vec{AC}) \cdot \vec{AD} \right|}{\left| \vec{AB} \times \vec{AC} \right|}$

$\vec{AB} = (-1, 1, 0)$

$\vec{AC} = (-3, -3, 2)$

$\vec{AD} = (2, 3, -4)$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ -3 & -3 & 2 \end{vmatrix} = 2\vec{i} - (-2)\vec{j} + 6\vec{k}$$

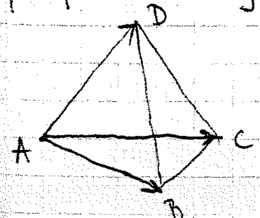
$$\vec{AB} \times \vec{AC} = (2, 2, 6)$$

$$(\vec{AB} \times \vec{AC})\ \vec{AD} = \begin{vmatrix} -1 & 1 & 0 \\ -3 & -3 & 2 \\ 2 & 3 & -4 \end{vmatrix} = 6 \cdot (-1) - 8 \cdot 1 + 0 = -14$$

$$(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = 2 \cdot 2 + 2 \cdot 3 + 6 \cdot (-4) = -14$$

$$H = \frac{|-14|}{\sqrt{4+4+36}} = \frac{14}{\sqrt{44}} = \frac{14}{2\sqrt{11}} = \frac{7}{\sqrt{11}} = \frac{7\sqrt{11}}{11}$$

2. Zapremina tetraedra je 5. Tri njegova tjemena su $A(2,1,-1)$, $B(3,0,1)$, $C(2,-1,3)$. Naći koordinate četvrtog tjemena D ako ono pripada Oy osi



$$D(0, b, 0)$$

$$V_t = \frac{1}{6} \left| (\vec{AB} \times \vec{AC}) \cdot \vec{AD} \right| = 5$$

$|(\vec{AB}\times\vec{AC})\cdot\vec{AD}| = 30$

$\vec{AB} = (1,-1,2)$

$\vec{AC} = (0,-2,4)$

$\vec{AD} = (-2, b-1, 1)$

$$(\vec{AB}\times\vec{AC})\cdot\vec{AD} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ -2 & b-1 & 1 \end{vmatrix} = (-2-4(b-1))\cdot 1 - 8\cdot(-1) + (-4)\cdot 2 = 2-4b$$

$|2-4b| = 30$

$2|1-2b| = 30$

$|1-2b| = 15$

$1^\circ \quad 1-2b = 15$

$2b = -14$

$b = -7$

$D(0,-7,0)$

$2^\circ \quad 1-2b = -15$

$2b = 16$

$b = 8$

$D(0,8,0)$

3. Dati su vektori $\vec{m} = (1,1,1)$ i $\vec{n} = (2,1,1)$. Vektor $\vec{p}$ sa osom Ox zaklapa ugao $\frac{\pi}{4}$, a sa osom Oy ugao $\frac{\pi}{3}$. Zapremina tetraedra nad vektorima $\vec{m}, \vec{n}$ i $\vec{p}$ je 2. Odredi vektor $\vec{p}$ ako on sa osom Oz zaklapa tup ugao.

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\alpha = \frac{\pi}{4}, \ \beta = \frac{\pi}{3}$

$\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{3} + \cos^2\gamma = 1$

$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$

$\frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1$

$\cos^2\gamma = \frac{1}{4}$

$\cos\gamma = \pm\frac{1}{2}$

$\gamma = \angle(\vec{p}, Oz)$ - tup ugao $\Rightarrow \cos\gamma < 0$

$\cos\gamma = -\frac{1}{2} \Rightarrow \gamma = \frac{2\pi}{3}$

$\vec{p}_0$ - jedinični vektor vektora $\vec{p}$

$\vec{p}_0 = (\cos\alpha, \cos\beta, \cos\gamma)$

$\vec{p}_0 = \left(\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

$\vec{p} = |\vec{p}|\cdot\vec{p}_0$

$|\vec{p}| = k, \ k > 0$

$\vec{p} = k\left(\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

$\vec{p} = \left(\frac{k\sqrt{2}}{2}, \frac{k}{2}, -\frac{k}{2}\right)$

$$V_t = \frac{1}{6} \left| (\vec{m} \times \vec{n}) \cdot \vec{p} \right|$$

$$V_t = 2$$

$$\left| (\vec{m} \times \vec{n}) \cdot \vec{p} \right| = 12$$

$$\vec{m} \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0 \cdot \vec{i} - (-1) \cdot \vec{j} + (-1) \cdot \vec{k}$$

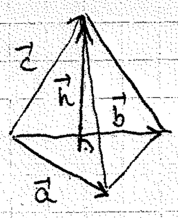
$$\vec{m} \times \vec{n} = (0, 1, -1)$$

$$(\vec{m} \times \vec{n}) \cdot \vec{p} = 0 + \frac{k}{2} + \frac{k}{2} = k$$

$$\left| (\vec{m} \times \vec{n}) \vec{p} \right| = 12 \Rightarrow |k| = 12$$

$$k = 12 \Rightarrow \boxed{\vec{p} = (6\sqrt{2}, 6, -6)}$$

4. Naći vektor visine tetraedra konstruisanog nad vektorima $\vec{a}, \vec{b}$ i $\vec{c}$.



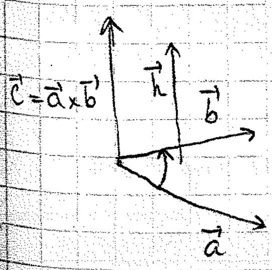
$\vec{h} = |\vec{h}| \cdot \vec{h}_0$, $\vec{h}_0$ - jedinični vektor vektora $\vec{h}$

$$V_t = \frac{1}{6} \left| (\vec{a} \times \vec{b}) \cdot \vec{c} \right|$$

$$V_t = \frac{1}{3} B \cdot |\vec{h}| = \frac{1}{3} \cdot \frac{1}{2} |\vec{a} \times \vec{b}| \cdot |\vec{h}|$$

$$|\vec{h}| = \frac{\left| (\vec{a} \times \vec{b}) \cdot \vec{c} \right|}{|\vec{a} \times \vec{b}|}$$

$$\left. \begin{array}{l} \vec{h} \perp \vec{a} \\ \vec{h} \perp \vec{b} \end{array} \right\} \vec{h} = \lambda (\vec{a} \times \vec{b}) \qquad \vec{h}_0 = \frac{1}{|\vec{a} \times \vec{b}|} (\vec{a} \times \vec{b})$$

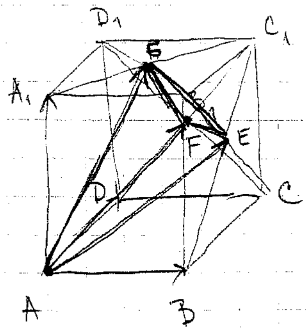


S obzirom na izabranu orijentaciju vektora $\vec{h}$, vektori $\vec{h}$ i $\vec{a} \times \vec{b}$ su istog smjera pa su im jedinični vektori jednaki

$$\vec{h} = \frac{\left| (\vec{a} \times \vec{b}) \cdot \vec{c} \right|}{|\vec{a} \times \vec{b}|} \cdot \frac{1}{|\vec{a} \times \vec{b}|} \vec{a} \times$$

$$\vec{h} = \frac{\left| (\vec{a} \times \vec{b}) \cdot \vec{c} \right|}{|\vec{a} \times \vec{b}|^2} \cdot \vec{a} \times \vec{b}$$

5. Tjeme paralelopipeda i centri tri njegove nasprамne strane čine tjemena tetraedra. Naći odnos zapremina paralelopipeda i tetraedra



$\vec{AB} = \vec{a}$ $\quad$ V - zapremina paralelopipeda

$\vec{AD} = \vec{b}$

$\vec{AA_1} = \vec{c}$ $\quad$ $V = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$

E - centar paralelograma BCC_1B_1

F - — " — DCC_1D_1

G - — " — $A_1B_1C_1D_1$

V_t - zapremina tetraedra AEFG

$$V_t = \frac{1}{6} |(\vec{AE} \times \vec{AF}) \cdot \vec{AG}|$$

$$\vec{AE} = \vec{AB} + \vec{BE} = \vec{a} + \frac{1}{2}\vec{BC_1} = \vec{a} + \frac{1}{2}(\vec{BC} + \vec{CC_1}) = \vec{a} + \frac{1}{2}(\vec{b} + \vec{c}) = \vec{a} + \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}$$

$$\vec{AF} = \vec{AD} + \vec{DF} = \vec{b} + \frac{1}{2}DC_1 = \vec{b} + \frac{1}{2}(\vec{DC} + \vec{CC_1}) = \vec{b} + \frac{1}{2}(\vec{a} + \vec{c}) = \frac{1}{2}\vec{a} + \vec{b} + \frac{1}{2}\vec{c}$$

$$\vec{AG} = \vec{AA_1} + \vec{A_1G} = \vec{c} + \frac{1}{2}\vec{A_1C_1} = \vec{c} + \frac{1}{2}(\vec{A_1B_1} + \vec{B_1C_1}) = \vec{c} + \frac{1}{2}(\vec{a} + \vec{b}) = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b} + \vec{c}$$

$$\vec{AE} \times \vec{AF} = (\vec{a} + \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}) \times (\frac{1}{2}\vec{a} + \vec{b} + \frac{1}{2}\vec{c}) =$$

$$= \vec{a} \times \vec{b} + \frac{1}{2}\vec{a} \times \vec{c} + \frac{1}{4}\vec{b} \times \vec{a} + \frac{1}{4}\vec{b} \times \vec{c} + \frac{1}{4}\vec{c} \times \vec{a} + \frac{1}{2}\vec{c} \times \vec{b} =$$

$$= \frac{3}{4}\vec{a} \times \vec{b} + \frac{1}{4}\vec{a} \times \vec{c} - \frac{1}{4}\vec{b} \times \vec{c}$$

$$(\vec{AE} \times \vec{AF}) \cdot \vec{AG} = (\frac{3}{4}\vec{a} \times \vec{b} + \frac{1}{4}\vec{a} \times \vec{c} - \frac{1}{4}\vec{b} \times \vec{c}) \cdot (\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b} + \vec{c}) =$$

$$= \frac{3}{4}(\vec{a} \times \vec{b}) \cdot \vec{c} + \frac{1}{8}(\vec{a} \times \vec{c}) \cdot \vec{b} - \frac{1}{8}(\vec{b} \times \vec{c}) \cdot \vec{a} =$$

$$= \frac{3}{4}(\vec{a} \times \vec{b}) \cdot \vec{c} + \frac{1}{8}(\vec{c} \times \vec{b}) \cdot \vec{a} - \frac{1}{8}(\vec{c} \times \vec{a}) \cdot \vec{b} =$$

$$= \frac{3}{4}(\vec{a} \times \vec{b}) \cdot \vec{c} + \frac{1}{8}(\vec{b} \times \vec{a}) \cdot \vec{c} - \frac{1}{8}(\vec{a} \times \vec{b}) \cdot \vec{c} =$$

$$= \frac{5}{8}(\vec{a} \times \vec{b})\vec{c} - \frac{1}{8}(\vec{a} \times \vec{b})\vec{c} = \frac{1}{2}(\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$V_t = \frac{1}{6}\left|\frac{1}{2}(\vec{a}\times\vec{b})\cdot\vec{c}\right| = \frac{1}{12}\left|(\vec{a}\times\vec{b})\cdot\vec{c}\right| = \frac{1}{12}V$$

6. Dat je tetraedar OABC zapremine $V=\sqrt{3}$. Poznato je da vektori $\vec{OA}$, $\vec{OB}$ i $\vec{OC}$ čine desno orijentisanu bazu i da važi $|\vec{OC}|=1$

$\measuredangle(\vec{OA},\vec{OB}) = \frac{\pi}{3}$, $\measuredangle(\vec{OA},\vec{OC}) = \measuredangle(\vec{OB},\vec{OC}) = \frac{\pi}{2}$

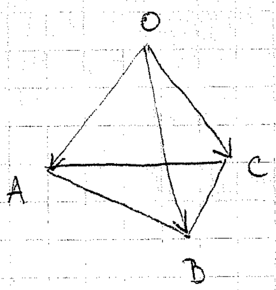
a) ako je $|\vec{OA}|=|\vec{OB}|$ izračunati $|\vec{OA}|$

b) odrediti realne brojeve α, β, γ takve da je $\vec{AB}\times\vec{AC} = \alpha\vec{OA}+\beta\vec{OB}+\gamma\vec{OC}$

$(\vec{OA}\times\vec{OB})\cdot\vec{OC} > 0 \quad (*)$

$(\vec{OB}\times\vec{OC})\cdot\vec{OA} > 0$

$(\vec{OC}\times\vec{OA})\cdot\vec{OA} > 0$



a) $V_t = \frac{1}{6}\left|(\vec{OA}\times\vec{OB})\cdot\vec{OC}\right|$

$V_t = \sqrt{3}$

$\left|(\vec{OA}\times\vec{OB})\cdot\vec{OC}\right| = 6\sqrt{3} \quad (**)$

$$\left.\begin{matrix}\vec{OC}\perp\vec{OA}\\ \vec{OC}\perp\vec{OB}\end{matrix}\right\} \Rightarrow \vec{OC} = \lambda(\vec{OA}\times\vec{OB})$$

$$\left|\,|\vec{OA}\times\vec{OB}|\,|\vec{OC}|\cos\overset{0 \text{ ili } \pi}{\measuredangle(\vec{OA}\times\vec{OB}),\vec{OC}}\right| = 6\sqrt{3}$$

$|\vec{OA}\times\vec{OB}|\,|\vec{OC}|\cdot 1 = 6\sqrt{3}$

$|\vec{OA}|\,|\vec{OB}|\sin\frac{\pi}{3} = 6\sqrt{3}$

$|\vec{OA}|^2\cdot\frac{\cancel{\sqrt{3}}}{2} = 6\cancel{\sqrt{3}}$

$|\vec{OA}|^2 = 12$

$|\vec{OA}| = \sqrt{12} = 2\sqrt{3}$

b) $\vec{AB}\times\vec{AC} = \underbrace{\alpha\vec{OA}+\beta\vec{OB}+\gamma\vec{OC}}$

$(\vec{OB}-\vec{OA})\times(\vec{OC}-\vec{OA}) = -\parallel-$

$\vec{OB}\times\vec{OC} - \vec{OB}\times\vec{OA} - \vec{OA}\times\vec{OC} =$

$= \alpha\vec{OA}+\beta\vec{OB}+\gamma\vec{OC} \quad /\cdot\vec{OA}$

$(\vec{OB}\times\vec{OC})\vec{OA} = \alpha|\vec{OA}|^2 + \beta\vec{OB}\cdot\vec{OA} + 0$

$(\vec{OB}\times\vec{OC})\vec{OA} = \alpha|\vec{OA}|^2 + \beta|\vec{OB}||\vec{OA}|\cos\cdot$

$\parallel$

$(\vec{OA}\times\vec{OB})\cdot\vec{OC} = 6\sqrt{3}$ iz $(*)$ i $(**)$

$12\alpha + 12\cdot\frac{1}{2}\beta = 6\sqrt{3}$

$12\alpha + 6\beta = 6\sqrt{3} \quad /:6$

$2\alpha+\beta = \sqrt{3} \quad (1)$

$$\vec{OB}\times\vec{OC} - \vec{OB}\times\vec{OA} - \vec{OA}\times\vec{OC} = \alpha\vec{OA} + \beta\vec{OB} + \gamma\vec{OC} \quad / \cdot \vec{OB}$$

$$-(\vec{OA}\times\vec{OC})\cdot\vec{OB} = \alpha\,\vec{OA}\cdot\vec{OB} + \beta\,|\vec{OB}|^2$$

$$(\vec{OC}\times\vec{OA})\cdot\vec{OB} = \alpha\,|\vec{OA}|\,|\vec{OB}|\cos\frac{\pi}{3} + \beta\,|\vec{OB}|^2$$

$$(\vec{OA}\times\vec{OB})\cdot\vec{OC} = \alpha\,|\vec{OA}|\,|\vec{OB}|\cos\frac{\pi}{3} + \beta\,|\vec{OB}|^2$$

$$\overset{6}{\cancel{12}}\cdot\frac{1}{2}\cdot\alpha + 12\beta = 6\sqrt{3}$$

$$6\alpha + 12\beta = 6\sqrt{3} \quad /:6$$

$$\alpha + 2\beta = \sqrt{3} \qquad (2)$$

Iz (1) i (2):
$$\begin{cases} 2\alpha + \beta = \sqrt{3} & / \cdot(-2) \\ \alpha + 2\beta = \sqrt{3} \end{cases}$$

$$+\begin{cases} -4\alpha - \cancel{2\beta} = -2\sqrt{3} \\ \alpha + \cancel{2\beta} = \sqrt{3} \end{cases}$$

$$-3\alpha = -\sqrt{3}$$

$$\boxed{\alpha = \frac{\sqrt{3}}{3}} \Rightarrow \beta = \sqrt{3} - \frac{2\sqrt{3}}{3}$$

$$\boxed{\beta = \frac{\sqrt{3}}{3}}$$

$$\vec{OB}\times\vec{OC} - \vec{OB}\times\vec{OA} - \vec{OA}\times\vec{OC} = \alpha\vec{OA} + \beta\vec{OB} + \gamma\vec{OC} \quad / \cdot \vec{OC}$$

$$-(\vec{OB}\times\vec{OA})\cdot\vec{OC} = \gamma\,|\vec{OC}|^2$$

$$(\vec{OA}\times\vec{OB})\cdot\vec{OC} = \gamma\cdot 1$$

$$6\sqrt{3} = \gamma$$

$$\boxed{\gamma = 6\sqrt{3}}$$

$$\vec{AB}\times\vec{AC} = \frac{\sqrt{3}}{3}\vec{OA} + \frac{\sqrt{3}}{3}\vec{OB} + 6\sqrt{3}\,\vec{OC}$$